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Minimal Irreversible Quantum Mechanics: Decoherence and Classical Equilibrium Limit

Mario Castagnino,[1] Edgard Gunzig,[2] and Roberto Laura[3]

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Using the Minimal Irreversible Quantum Mechanics formalism, it is demonstrated that the quantum regime can be considered as the transient phase while the final classical equilibrium regime is the permanent state. A basis where exact matrix decoherence appears for these final states is found. The appearance of a classical universe in quantum gravity models is the cosmological version of this problem.

1. INTRODUCTION

In ref. 1, following the development of a new formalism that we call Minimal Irreversible Quantum Mechanics, we studied the *relation* of the *state vectors* ρ of a closed isolated quantum system (which belong to a set of states $\mathcal{S}$) to the *observables* O within this closed system (which belong to a space of observables $\mathbb{O}$). We consider that the essence of this relation is the mean value of an observable O in a state ρ, which is given by the equation

$$\langle O\rangle_\rho = \mathrm{Tr}(\rho O) = (\rho|O) \tag{1.1}$$

In fact, what we actually measure is this kind of average, since we cannot either measure the state ρ directly or measure with infinite precision [2]. Moreover, these averages can be considered, as in the r.h.s. of Eq. (1.1), the result of a linear functional $(\rho| \in \mathcal{S}$ (a convex set) acting on a vector $|O) \in \mathbb{O}$, and therefore we can say that $\mathcal{S} \subset \mathbb{O}'$, with $\mathbb{O}'$ the dual of space $\mathbb{O}$. While for the usual states (mixed or pure) we can use $\mathrm{Tr}(\rho O)$, they are *generalized*

[1] Instituto de Astronomía y Física del Espacio, Casilla de Correos 67, 1428 Buenos Aires, Argentina.
[2] Service de Physique, Brussels Free University, CP 231, Campus Plaine, 1050 Brussels, Belgium.
[3] Departamento de Física, F.C.E.I.A., Universidad Nacional de Rosario. Instituto de Física de Rosario, CONICET-UNR, 2000 Rosario, Argentina.

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states that can be defined as the functional $(\rho|O)$ as explained in ref. 1. Many results were obtained using this mathematical formalism (see, e.g., refs. 1, 3, 4). On the other hand, the appearance of a classical universe in quantum gravity models is the cosmological version of the problem we are discussing [5]. In this paper we will use the formalism of ref. 1 to explain the phenomenon of decoherence which, combined with the disappearance of the uncertainty relations, in the limit $\hbar \to 0$, gives rise to the classical final equilibrium state. To solve these problems the usual technique is coarse-graining. But in our method we will consider not only the coarse-graining average, but *all* possible averages made using the observables of the space $\mathbb{O}$; thus we are generalizing the coarse-graining idea. In fact, among the observables of $\mathbb{O}$ there are some that, from the density matrix ρ, take into account only some component ρ_r, the so-called relevant part of ρ, and neglect completely the complementary component ρ_i, the so-called irrelevant part of ρ, i. e., they measure the properties of what it is considered as the "system" (contained in ρ_r) and neglect or average the properties of the "environment" (contained in ρ_i). But we will consider not only this kind of observable, but all observables in $\mathbb{O}$. Therefore the interplay of observables and states will take the role of the coarse-graining in this paper [see also Eq. (2.16)]. With this strategy we can obtain all the old results, but we will also find some new ones.

We will use this method to prove that a large class of quantum systems evolve from the quantum phase to classical final equilibrium: precisely, from a phase where we must use the laws of (statistical) quantum mechanics to a phase where we must use the laws of statistical classical mechanics [2]. The main characteristic of the quantum laws are as follows:

1. The non-Boolean nature of the way to find the probability of two exclusive events (considering that this probability is the square modulus of the sum of their amplitudes and not the sum of the probabilities).

2. The uncertainty relations.

In the quantum to classical evolution the first characteristic disappears (and the Boolean way of adding probabilities is established) by a process known as *decoherence* and the uncertainty relations can be neglected in the limit $\hbar \to 0$. Then we can use the laws of classical statistical mechanics.

Four observations are in order:

(i) Using our language, the generalized idea of decoherence can be stated in the following way: At the quantum level the average (1.1) reads

$$\langle O \rangle_\rho^{(q)} = \sum_{\omega,\omega'} \rho_{\omega\omega'} O_{\omega'\omega} \tag{1.2}$$

where $\rho_{\omega\omega'}$ and $O_{\omega\omega'}$ are the components of the operators ρ and O, respectively (and where the index ω symbolizes all the necessary indices; to fix the ideas, let us consider the index ω as the eigenvalue of the operator O that, for

simplicity, we take as nondegenerate). Equation (1.2) can be considered as the average of some quantities $O_{\omega\omega'}$ weighted by some generalized correlations $\rho_{\omega\omega'}$ (since the $\rho_{\omega\omega'}$ are probabilities, but the $\rho_{\omega\omega'}$, with $\omega \neq \omega'$, are quantum correlations). On the other hand, at the classical level we also have some quantities O_ω that correspond to a set $\{\omega\}$ of the exhaustive and exclusive alternatives, each one with a (Boolean) probability p_ω of measure ω for the observable O. The corresponding classical weighted average is

$$\langle O \rangle_\rho^{(cl)} = \sum_\omega p_\omega O_\omega \tag{1.3}$$

where $\Sigma_\omega p_\omega = 1$. The transition for the quantum phase to the classical one is therefore

$$\sum_{\omega,\omega'} \rho_{\omega\omega'} O_{\omega\omega'} \to \sum_\omega p_\omega O_\omega \tag{1.4}$$

at least for some O which belong to a preferred subspace of $\mathbb{O}$ (i.e., to a subspace expanded by a complete set of commuting observables, CSCO, that we will define below; the eigenbasis of this set will be the so-called *final pointer basis*). If in (1.4) we take $\rho_{\omega\omega} = p_\omega$ and $O_{\omega\omega} = O_\omega$, these matrices must become diagonal in the final pointer basis (such privileged bases must exist since only the unit matrix I is a diagonal in any basis). This is the essence of the transition, since the relation above will be valid for all observables of the CSCO and we will have

$$\langle O \rangle_\rho^{(q)} \to \langle O \rangle_\rho^{(cl)} \tag{1.5}$$

If this transition takes place, Boolean logic is established in the statistical classical system if we perform the measurement with the observables of the preferred CSCO. In the usual parlance we will then say that the density matrices that contain quantum interference terms become diagonal, in such away that these interferences are suppressed. Then the quantum way to find probabilities of exclusive and exhaustive alternatives, i.e., adding the corresponding amplitudes and computing the norm, becomes the classical Boolean way: just adding the probabilities. Moreover, the uncertainty relations will disappear, when $\hbar \to 0$, precisely when the characteristic dimension of the system makes $\hbar$ a negligible quantity.

(ii) In this paper decoherence is essentially studied in systems with continuous spectrum. The case of discrete spectrum, and the causes of decoherence in this case, are discussed in Section 2.3.

(iii) In the case of the continuous spectrum the essence of the method is the following: If we call $\nu = \omega - \omega'$, the $\rho_{\omega\omega'}$ of Eq. (1.2) is a function $\rho(\nu, \ldots)$. Then the time limit of its evolution is given by the Riemann–Lebesgue theorem, which prescribes that

$$\lim_{t\to\infty} \int_{-a}^{a} e^{-i\nu t}\rho(\nu,\ldots)\, d\nu = 0 \tag{1.6}$$

if $\rho(\nu, \ldots)$ is integrable. So all diagonal terms ($\nu = 0$) or off-diagonal terms ($\nu \neq 0$) vanish. Therefore this theorem cannot be used as a computation method in the case of continuous spectrum. Nevertheless if we consider the problem within a cube L^3, define $\rho_{\omega\omega'}$ there, and make $L \to \infty$, we can show that a singular structure appears for $\rho(\nu, \ldots)$:

$$\rho(\nu, \ldots) = \rho_S(\nu, \ldots)\delta(\nu) + \rho_R(\nu, \ldots) \tag{1.7}$$

where $\rho_S(\nu, \ldots)$ and $\rho_R(\nu, \ldots)$ are regular integrable functions. Now Eq. (1.6) reads

$$\lim_{t\to\infty} \int_{-\alpha}^{a} e^{-i\nu t}\, \rho(\nu, \ldots)\, d\nu = \rho_S(0, \ldots) \tag{1.8}$$

and the diagonal term remains. The method introduced in ref. 1 is just designed to deal with the singular δ function in Eq. (1.7).

(iv) Before the classical equilibrium limit is reached usually the system goes through a "classical phase" out of classical equilibrium. In this paper we mostly study the final equilibrium state, but we believe that our method can be generalized to cover the classical phase. Moreover, we believe that understanding the equilibrium limit, in the clearest and most concise way, will enhance the chances to understand the much more difficult problem of the classical phase.

Following these ideas, we will see (Section 2), using the Riemann–Lebesgue theorem, that the required transition (1.5) takes place in all closed systems endowed with a continuous spectrum (like the classical mixing systems). More general cases will be considered in Section 2.3.

In Section 3 we reach the principal aim of the formalism, which is to create a bridge between quantum and classical mechanics, precisely between quantum mechanics and classical statistical mechanics. Let us consider a system whose quantum state is defined by a density matrix ρ, and a set of classical trajectories in phase space labeled by some constants $n, l_1, \ldots, l_N, a_1, \ldots, a_N$, where n corresponds to the energy, $l_i, \ldots, l_N$ to other dynamical momentum variables, and $a_1, \ldots, a_N$ to configuration variables. The aim of the theory is:

1. To transform the matrix ρ into a classical density function in phase space $\rho(q, p)$ when $\hbar \to 0$.

2. To decompose $\rho(q, p)$ as

$$\rho(q, p) = \sum_{n,l_1,\ldots,l_N,a_1,\ldots,a_N} p_{n,l_1,\ldots,l_N,a_1,\ldots,a_N}\rho_{n,l_1,\ldots,l_N,a_1,\ldots,a_N}(q, p) \tag{1.9}$$

where x and p are the position and momentum coordinates and the classical

densities $\rho_{n,l_1,\ldots,l_N,a_1,\ldots,a_N}(q, p)$ would correspond to each classical trajectory[4] (in the classical sense that it is peaked in the trajectory and thus it vanishes from the near neighborhood of the trajectory to the far zones of the phase space) and $p_{n,l_1,\ldots,l_N,a_1,\ldots,a_N}$ is the probability of each trajectory.

We will obtain the following results:

1. $\rho(q, p)$ will be the Wigner function corresponding to ρ.

2. $\rho_{n,l_1,\ldots,l_N,a_1,\ldots,a_N}(q, p)$ will be the Wigner functions of the wave packets going along the classical trajectories labelled by the constant of the motion $n, l_1, \ldots, l_N$, and passing through the initial point of coordinates $a_1, \ldots, a_N$.

We will see that all this happens after a convenient decoherence time and we will obtain the last expansion in Eq. (3.19), and therefore what we consider the best bridge between classical and quantum concepts (see ref. 6 for a very similar conclusion).

We draw our main conclusions in Section 5.

2. DECOHERENCE

2.1. Decoherence in the Energy

Let us consider a closed and isolated quantum system with $N + 1$ dynamical variables and a Hamiltonian

$$H = H_0 + \lambda W \tag{2.1}$$

where λ is a coupling constant (for some considerations we will take $\lambda \ll 1$, so in this case we will deal with underdamped systems, otherwise the formalism will be general). Let us suppose that H_0, the free Hamiltonian, has a spectrum with a discrete part (corresponding, e.g., to several bound states) and a continuous part (corresponding, e.g., to scattering states). Let us further suppose that the interaction λW produces the decay of all the bound states but (eventually) one: the ground state (we will consider the case of more than one bounded undecayed state in Section 2.3). So the discrete part of the spectrum of H has only one value ω_0 and the continuous spectrum is $0 \leq \omega < \infty$. (How the discrete spectrum behaves in the continuous limit can be seen in refs. 7 and 8). Eventually we will give the collective name n to both ω_0 and ω.

We will discuss decoherence, elimination of the uncertainty relations, and the outcome of the classical realm in this quite general model. Let us begin with decoherence.

[4] The dimension of the phase space considered is $2(N + 1)$. Then there are $(N + 1)$ momenta and $(N + 1)$ coordinates. So $N + 1$ is the number of parameters necessary to label the momenta of the classical space-time trajectories, and N the number necessary to label the origins of the trajectories.

To obtain the H-Hamiltonian eigenbasis of the Hilbert space $\mathcal{H}$ we can consider the Hamiltonian of Eq. (2.1) and construct the basis where it is diagonal, which we will call $\{|\omega\rangle\}$ (ω is the energy eigenvalue). It can be, e.g., the Lippmann–Schwinger basis [9] or any other basis that diagonalizes the Hamiltonian as

$$H = \omega_0|\omega_0\rangle\langle\omega_0| + \int_0^\infty \omega|\omega\rangle\langle\omega|\, d\omega \tag{2.2}$$

where $|\omega_0\rangle$ is the ground state. From this expression we can deduce that the most general observable that we can consider in our model reads

$$\begin{aligned} O = {} & O_0|\omega_0\rangle\langle\omega_0| + \int_0^\infty O_{0\omega}|\omega_0\rangle\langle\omega|\, d\omega + \int_0^\infty O_{\omega 0}|\omega\rangle\langle\omega_0|\, d\omega \\ & + \int_0^\infty O_\omega|\omega\rangle\langle\omega|\, d\omega + \int_0^\infty \int_0^\infty O_{\omega\omega'}|\omega\rangle\langle\omega'|\, d\omega\, d\omega' \end{aligned} \tag{2.3}$$

where the functions $O_{\omega 0}$, $O_{\omega 0}$, O_ω, $O_{\omega\omega'}$ are ordinary functions (these functions must have certain mathematical properties in order to develop the theory; these properties are listed in ref. 1). Namely, the most general observable must have a singular component (the fourth term of the r.h.s. of the last equation) and a regular part (all the other terms). If the singular term were missing, the Hamiltonian (2.2) would not belong to the space of the chosen observables [1]. We will say that these observables belong to a space $\mathbb{O}$. This space has the *basis* $\{|\omega_0), |\omega_0, \omega), |\omega, \omega_0), |\omega), |\omega, \omega')\}$:

$$\begin{gathered} |\omega_0) = |\omega_0\rangle\langle\omega_0|, \qquad |\omega_0, \omega) = |\omega_0\rangle\langle\omega|, \qquad |\omega, \omega_0) = |\omega\rangle\langle\omega_0| \\ |\omega) = |\omega\rangle\langle\omega|, \qquad |\omega, \omega') = |\omega\rangle\langle\omega'| \end{gathered} \tag{2.4}$$

The quantum states ρ are measured by the observables just defined, computing the mean values of these observables in the quantum states, i.e., in the usual notation, $\langle O\rangle_\rho = \mathrm{Tr}(O\rho)$ [2]. These mean values, generalized as in ref. 1, can be considered as linear functionals ρ (mapping the vectors O on the real numbers), which we can call $(\rho|O)$ [10]. In fact, this is a generalization of the usual mean value definition. Then $\rho \in \mathcal{S} \subset \mathbb{O}'$, where $\mathcal{S}$ is a convenient convex set contained in $\mathbb{O}'$, the space of linear functional over $\mathbb{O}$ [11, 12]. The basis of $\mathbb{O}'$ (which can also be considered as the *cobasis* of $\mathbb{O}$) is $\{(\omega_0|, (\omega_0, \omega|, (\omega, \omega_0|, (\omega|, (\omega, \omega'|\}$ defined as functionals by the equations

$$\begin{gathered} (\omega_0|\omega_0) = 1, \qquad (\omega_0, \omega|\omega_0, \omega') = \delta(\omega - \omega'), \\ (\omega_0, \omega|\omega_0, \omega') = \delta(\omega - \omega') \end{gathered} \tag{2.5}$$

$$(\omega|\omega') = \delta(\omega - \omega'), \qquad (\omega,\omega''|\omega', \omega''') = \delta(\omega - \omega')\delta(\omega'' - \omega''')$$

and all other $(\cdot|\cdot)$ are zero. In particular, we can define a functional $(|\omega_0\rangle\langle\omega_0|$ as

$$(\omega_0|O) = \langle\omega_0|O|\omega_0\rangle = \mathrm{Tr}(|\omega_0\rangle\langle\omega_0|O) = (|\omega_0\rangle\langle\omega_0\|O) \tag{2.6}$$

for any $O \in \mathbb{O}$: Then we can verify that

$$(\omega_0| = |\omega_0\rangle\langle\omega_0| \tag{2.7}$$

But

$$(\omega| \neq |\omega\rangle\langle\omega| \tag{2.8}$$

and $(\omega|$ can be considered only as a functional, being a typical generalized state. Therefore a generic quantum state reads

$$\rho = \rho_0(\omega_0| + \int_0^\infty \rho_{\omega 0}\,(\omega, \omega_0|\, d\omega + \int_0^\infty \rho_{0\omega}\,(\omega_0, \omega|\, d\omega + \int_0^\infty \rho_\omega(\omega|\, d\omega + \int_0^\infty \int_0^\infty \rho_{\omega\omega'}(\omega, \omega'|\, d\omega\, d\omega' \tag{2.9}$$

where ρ_0, ρ_ω are real and ≥ 0, $\rho^*_{0\omega} = \rho_{\omega 0}$, and $\rho^*_{\omega\omega'} = \rho_{\omega'\omega}$ (they must also have other properties listed in ref. 1). The time evolution of the quantum state ρ reads

$$\rho(t) = \rho_0(\omega_0| + \int_0^\infty \rho_{\omega 0} e^{i(\omega-\omega_0)t}(\omega, \omega_0|\, d\omega + \int_0^\infty \rho_{0\omega} e^{i(\omega_0-\omega)t}(\omega_0, \omega|\, d\omega + \int_0^\infty \rho_\omega(\omega|\, d\omega + \int_0^\infty \int_0^\infty \rho_{\omega\omega'} e^{i(\omega-\omega')t}\,(\omega, \omega'|\, d\omega\, d\omega' \tag{2.10}$$

As we only measure mean values of observables in quantum states, i.e.,

$$\langle O\rangle_{\rho(t)} = (\rho(t)|O) = (\rho|O(t)) = \rho_0 O_0 + \int_0^\infty \rho_{0\omega} O_{\omega 0} e^{i(\omega_0-\omega)t}\, d\omega + \int_0^\infty \rho_{\omega 0} O_{0\omega} e^{i(\omega-\omega_0)t}\, d\omega + \int_0^\infty O_\omega \rho_\omega\, d\omega + \int_0^\infty \int_0^\infty O_{\omega'\omega}\rho_{\omega\omega'} e^{i(\omega-\omega')t}\, d\omega\, d\omega \tag{2.11}$$

using the Riemann–Lebesgue theorem, we obtain the weak limit, for all $O \in \mathbb{O}$,

$$\lim_{t\to\infty} \langle O\rangle_{\rho(t)} = \langle O\rangle_{\rho_*} \tag{2.12}$$

where we have introduced the diagonal asymptotic or equilibrium operator

$$\rho_* = \rho_0(\omega_0| + \int_0^\infty \rho_\omega(\omega| \, d\omega \tag{2.13}$$

Therefore, in a weak sense (for the classical case see ref. 13) we have

$$\lim_{t\to\infty} \rho(t) = \rho_* \tag{2.14}$$

Thus, any quantum state goes weakly to a diagonal state, precisely to the state $(\omega_0|$ for the discrete spectrum and to a linear combination of the diagonal states $(\omega|$ of the continuous one [there are no off-diagonal terms in of the states since the of diagonal states $(\omega_0, \omega|$, $(\omega, \omega_0|$, or $(\omega, \omega'|$ are not present in them]. This is the case if we observe and measure the system evolution with *any possible observable of space* $\mathbb{O}$. Then, from the observational (or generalized coarse-graining) point of view, we have decoherence of the energy levels, even that, from the strong limit (fine-graining) point of view the off-diagonal terms never vanish, they just oscillate, since we cannot directly use the Riemann–Lebesgue theorem in the operator equation (2.10). So we have obtained a diagonal matrix in the energy.

Some observations are in order:

(i) The real existence of the two singular parts of O and ρ is assured by the physics of the problem. The singular part of the observables is just a necessary generalization of the singular part of the Hamiltonian, which has a singular part $|\omega)$ [Eq. (2.2)]. The singular part of the states is necessary, since the final state must have a diagonal singular part $(\omega|$ produced by the decoherence process [Eq. (2.13)].[5] The trace of the product of the these two parts is just the natural generalization to the continuous trace of the product of two finite-dimensional matrices [Eq. (2.11); see also ref. 1].

(ii) From Eq. (2.12) we can see again that what we are doing is just a generalized version of coarse-graining. In fact, let us define a projector on the "relevant part" of the system as

$$P = \sum_i |\rho_i)(O_i| \tag{2.15}$$

where $|\rho_i) \in \mathcal{S}$, $|O_i) \in \mathbb{O}$, and $(O_i|\rho_j) = \delta_{ij}$.[6] Then from Eq. (2.12) we have

$$\lim_{t\to\infty} P|\rho) = \lim_{t\to\infty} \sum_i |\rho_i)(O_i|\rho(t)) = \sum_i |\rho_i)(O_i|\rho_*) = P|\rho_*) \tag{2.16}$$

Thus the weak limit (2.12) can be considered as the limit of $P|\rho)$, the relevant

[5] In many case final equilibrium states are singular, e.g., in the Baker's transformation the final equilibrium state can be considered only as a functional.

[6] We have inverted the position of the O and the ρ in order to obtain the usual equation with the P acting from the left.

part of $|\rho)$, which goes to the relevant part of the equilibrium state through Eq. (2.16).

2.2. Decoherence in the Other "Momentum" Dynamical Variables

Having established the decoherence in the energy levels, we must consider the decoherence in the other dynamical variables O_i of the CSCO where we are working, which up to now have not been taken into account. We will call these variables "momentum variables." For the sake of simplicity we will consider that the spectra of these dynamical variables are discrete. The diagonal component of Eq. (2.10) (which is equal to ρ_*) is time independent, thus it is impossible that a different decoherence process would take place in this component to eliminate the off-diagonal terms in the other dynamical variables. Therefore, the only thing to do is to try find if there is a basis where these diagonal terms vanish at any time after the equilibrium is reached, and therefore there is a perfect and complete decoherence. This basis in fact exists, it is constant in time, and it is known as the *final pointer basis*.

Let $(H, O_1, \ldots, O_N)$ be the CSCO and $\{|\omega_0, m_1, \ldots, m_N\rangle, |\omega, m_1, \ldots, m_N\rangle\}$ be the basis that we are using to make our calculations. Thus

$$\begin{aligned} H|\omega_0, m_1, \ldots, m_N\rangle &= \omega_0|\omega_0, m_1, \ldots, m_N\rangle \\ H|\omega, m_1, \ldots, m_N\rangle &= \omega|\omega, m_1, \ldots, m_N\rangle \\ O_i|\omega_0, m_1, \ldots, m_N\rangle &= m_i|\omega_0, m_1, \ldots, m_N\rangle \\ O_i|\omega, m_1, \ldots, m_N\rangle &= m_i|\omega, m_1, \ldots, m_N\rangle \end{aligned} \tag{2.17}$$

Then we can add $m_1, \ldots, m_N$ to each symbol of the basis of Eq. (2.4), e.g.,

$$\begin{aligned} |\omega_0, m_1, \ldots, m_N, m'_1, \ldots, m'_N) &= |\omega_0, m_1, \ldots, m_N\rangle\langle\omega_0, m'_1, \ldots, m'_N| \\ |\omega, m_1, \ldots, m_N, m'_1, \ldots, m'_N) &= |\omega, m_1, \ldots, m_N\rangle\langle\omega, m'_1, \ldots, m'_N|, \quad \text{etc.} \end{aligned} \tag{2.18}$$

Also we can add the symbols $m_1, \ldots, m_N$ to the cobasis $\{(\omega_0|, (\omega_0, \omega|, (\omega, \omega_0|, (\omega|, (\omega, \omega'|\}$ as $(\omega_0, m_1, \ldots, m_N, m'_N, \ldots, m'_N|$, $(\omega, m_1, \ldots, m_N, m'_1, \ldots, m'_N|$, etc., defined through the natural generalization of Eq. (2.5).

Then the equilibrium diagonal operator reads[7]

$$\rho_* = \sum_{m_1,\ldots,m_N,m'_1,\ldots,m'_N} [\rho^{(\omega_0)}_{m_1,\ldots,m_N,m'_1,\ldots,m'_N}(\omega_0, m_1, \ldots, m_N, m'_1, \ldots, m'_N| + \int_0^\infty \rho^{(\omega)}_{m_1,\ldots,m_N,m'_1,\ldots,m'_N}(\omega, m_1, \ldots, m_N, m'_1, \ldots, m'_N|\, d\omega] \tag{2.19}$$

[7] For simplicity, we have considered that the indices $m_i, \ldots, m_N$ are discrete, so the Σ in Eq. (2.12) is only an infinite sum.

Now, from the properties of the components on any ρ [see below Eq. (2.9)] we have

$$(\rho^{(\omega_0)}_{m_1,\ldots,m_N,m'_1,\ldots,m'_N})^* = \rho^{(\omega_0)}_{m'_1,\ldots,m'_N,m_1,\ldots,m_N}$$

$$(\rho^{(\omega)}_{m_1,\ldots,m_N,m'_1,\ldots,m'_N})^* = \rho^{(\omega)}_{m'_1,\ldots,m'_N,m_1,\ldots,m_N} \tag{2.20}$$

Therefore there is a basis $\{(\omega_0, l_1, \ldots, l_N|, (\omega, l_1, \ldots, l_N|\}$ that diagonalizes the matrix ρ_* as

$$\rho_* = \sum_{n,l_1,\ldots,l_N} [\rho^{(\omega_0)}_{l_1,\ldots,l_N}(\omega_0, l_1, \ldots, l_N| + \int_0^\infty \rho^{(\omega)}_{l_1,\ldots,l_N}(\omega, l_1, \ldots, l_N|\, d\omega] \tag{2.21}$$

where $\rho^{(\omega_0)}_{l_1,\ldots,l_N} \geq 0$ and $\rho^{(\omega)}_{l_1,\ldots,l_N} \geq 0$. The basis $\{(\omega_0, l_1, \ldots, l_N|, (\omega_0, \omega, l_1, \ldots, l_N|, (\omega, \omega_0, l_1, \ldots, l_N|, (\omega, l_1, \ldots, l_N|, (\omega, \omega', l_1, \ldots, l_N|, (\omega, \omega', l_1, \ldots, l_N, l'_1, \ldots, l'_N|\}$ is the final pointer cobasis, namely the basis where ρ_* is diagonal in all the dynamical variables. As the last diagonalization was done in the discrete indices, $m_1, \ldots, m_N \to l_1, \ldots, l_N$, while ω_0 and ω were untouched, the pointer basis for the observables is $\{|\omega_0, l_1, \ldots, l_N), |\omega_0, \omega, l_1, \ldots, l_N), |\omega, \omega_0, l_1, \ldots, l_N), |\omega, l_1, \ldots, l_N), |\omega, \omega', l_1, \ldots, l_N)\}$, where

$$|\omega_0, l_1, \ldots, l_N) = |\omega_0, l_1, \ldots, l_N\rangle\langle\omega_0, l_1, \ldots, l_N|$$

$$|\omega, l_1, \ldots, l_N) = |\omega, l_1, \ldots, l_N\rangle\langle\omega, l_1, \ldots, l_N|, \quad \text{etc.} \tag{2.22}$$

which satisfies Eq. (2.5) with respect to the final pointer cobasis for the states. Now we can define the *exact final pointer observables* [14]

$$P_i = \sum_{n,l_1,\ldots,l_N} [P^{(i,\omega_0)}_{l_1,\ldots,l_N} |\omega_0, l_1, \ldots, l_N\rangle\langle\omega_0, l_1, \ldots, l_N| + \int_0^\infty P^{(i,\omega)}_{l_1,\ldots,l_N} |\omega, l_1, \ldots, l_N\rangle\langle\omega, l_1, \ldots, l_N|\, d\omega] \tag{2.23}$$

As H and the P_i are diagonal in the basis $\{|\omega_0, l_1, \ldots, l_N\rangle, |\omega, l_1, \ldots, l_N\rangle\}$, the CSCO $\{H, P_1, \ldots, P_N\}$ is precisely the complete set of commuting observables (CSCO) related to this basis, where ρ_* is diagonal in the corresponding cobasis for the states. For simplicity we define the operators P_i such that the $P^{(i,\omega_0)}_{l_1,\ldots,l_1,\ldots,l_N} = l_i$ and the $P^{(i,\omega)}_{l_1,\ldots,l_1,\ldots,l_N} = l_i$; thus[8]

$$P_i|\omega_0, l_1, \ldots, l_N\rangle = l_i|\omega_0, l_1, \ldots, l_N\rangle$$

$$P_i|\omega, l_1, \ldots, l_N\rangle = l_i|\omega, l_1, \ldots, l_N\rangle \tag{2.24}$$

[8] For simplicity, we consider that the indices l_i, like the m_i, are discrete, so we are supposing that the P_i have a discrete spectrum, e.g., they are angular momenta.

So $\{|\omega_0, I_1, \ldots, I_N\rangle, |\omega, I_1, \ldots, I_N\rangle\}$ is the observers pointer basis where there is perfect decoherence in the corresponding states cobasis. Moreover, the states $(\omega_0, I_1, \ldots, I_N| = |\omega_0, I_1, \ldots, I_N\rangle\langle\omega_0, I_1, \ldots, I_N|$ or $(\omega, I_1, \ldots, I_N| = |\omega, I_1, \ldots, I_N\rangle\langle\omega, I_1, \ldots, I_N|$ are constants of the motion and therefore these exact pointer observables have a constant statistical entropy and will be "at the top of the list" of Zurek's "predictability sieve" [14]. The final pointer basis is therefore defined by the dynamics of the model and by the considered quantum state.

Therefore:

(i) Decoherence in the energy is produced by the time evolution.

(ii) Decoherence in the other dynamical variables can be seen if we choose an adequate basis, namely the final pointer basis.

Essentially we have given a partial answer, for this kind of model, to the fundamental question of Gell-Mann and Hartle [15] (precisely, only an answer in the case $t \to \infty$): For each H and in each initial state ρ there is only one final pointer basis and therefore only one "quasiclassical domain or realm" [16].[9]

Our main result is Eq. (2.21): When $t \to \infty$, then $\rho(t) \to \rho_*$ and in this state the dynamical variables $H, P_1, \ldots, P_N$ are well defined. Therefore the eventual conjugate variables to these momentum variables (namely, configuration variables if they exist) are completely undefined. Then ρ_* is homogeneous in these configuration variables, in the sense that it is $\rho_*(H, P_1, \ldots, P_N)$, because $H, P_1, \ldots, P_N)$ are a CSCO, and ρ_* "commutes"[10] with all the $H, P_1, \ldots, P_N$, and therefore it is not a function of the eventual configuration variables.

2.3. Decoherence Characteristic Decaying Time, the Permanent Quantum States Case, and the Role of the Environment

From the preceding section we may have the feeling that the process of decoherence must be found in all physical systems, and therefore all of them eventually would become classical when $\hbar \to 0$. This is not so and there are two reasons:

(i) Characteristic decay times can be computed using analytic continuation techniques, as in ref. 1. For example, in particular models we can find the characteristic times for the system (e.g., an oscillator) and the field (e.g., the environment or bath) as below Eq. (56) of ref. 1. If the maximal characteristic time γ^{-1} is very large, even if theoretically the decoherence process always takes place, it will be so slow that the system will behave as

[9] But this unique consistent set depends of course, on the chosen space of observable $\mathbb{O}$ (see more in Appendix C).

[10] Namely, they commute in the functional sense that $([H, \rho_*]|A) = (\rho_*|[A, H]) = 0$ for all $A \in \mathbb{O}$.

a quantum one for a long time. Then there will not be a measurable decoherence. It may also happen that more than one of the γ would be zero. Then we will have more than one possible ground state (bound states), as considered in next.

(ii) Another way to understand the existence of permanent quantum systems is to consider the case when the complete Hamiltonian H has more than one bound state, let us say n (or when the spectrum is discrete). Then the first term of the r.h.s. of Eq. (2.11) becomes

$$\sum_{ij} O_{ij}\rho_{ji}\, e^{-i(\omega_i-\omega_j)t} = \sum_{i} O_{ii}\rho_{ii} + \sum_{i\neq j} O_{ij}\rho_{ji}\, e^{-i(\omega_i-\omega_j)t} \tag{2.25}$$

where $i, j = 1, \ldots, n$, and as the second term of the r.h.s. does not vanish, decoherence does not take place. This is the case of a theoretical atom, not coupled to the electromagnetic field, where the electrons remain forever in their excited states and never decay. Then the atom never goes to a decohered state. But if the atom is coupled to an electromagnetic field (usually called the "environment"), there will be only one ground state, $n = 1$, the second term of the r.h.s. of Eq. (2.25) will be absent, and decoherence will occur.

In fact, in many examples the role of the "environment" is just to introduce a continuous spectrum to be coupled in such a way that only one bound state remains and decoherence is complete. In other cases fluctuations (or imperfections) of continuous nature take the role of the continuous spectrum and produce an average and make the diagonal term disappear. This is the case of the spin recombination experiment ref. 2, p. 180) in a single crystal interferometer.

(iii) More generally, the components ρ_{nm} of a state can be found using the observables [2]

$$A_{nm} = \tfrac{1}{2}\left(|n\rangle\langle m| + |m\rangle\langle n|\right), \qquad B_{nm} = \tfrac{i}{2}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{2.26}$$

Using only observables from a subset $\Omega \in \mathbb{O}$, we can only found some ρ_{nm}, e.g., those constructed with the eigenvalues $|n\rangle$ and $|m\rangle$ of H that eventually expand the space Ω. Then if we consider only the observables of Ω, the components ρ_{nm} related with these observables can become classical because their decoherence time is small, while the other components remain quantum because they have a larger decoherence time. Then we will have a system which is partially classical and partially quantum (which in fact is the case of the universe where there are both classical and quantum phenomena).

3. THE CLASSICAL $\rho_*^{(cl)}(q, p)$

3.1. Expansion in Sets of Classical Motions

In this section we will use the Wigner integral, which introduces an isomorphism between quantum observables and states O and ρ and their

classical analogues O_W and $\rho_W = \rho^{(cl)}$ (which, in general, it is not positive definite). In fact, we have that

$$(\rho_W|O_W) = \int\int \rho_W(q, p)O_W(q, p)\, dp\, dq = \mathrm{Tr}(\rho O) \tag{3.1}$$

Therefore we can consider generalized classical densities ρ_W as functionals over classical observables O_W and also generalize the previous equation to

$$(\rho|O) = (\rho^{(cl)}|O_W) \tag{3.2}$$

In this way some singular classical densities, like the Dirac deltas that we will use below, get a rigorous meaning.

We will prove that the distribution function $\rho_*^{(cl)}(q, p)$ that corresponds to the density matrix ρ_* via the Wigner integral [17] is simply a positive-definite function of the classical constant of the motion, in our case $H_W(q, p)$, $P_{W1}(q, p), \ldots, P_{WN}(q, p)$, obtained in the same way from the corresponding quantum operator; precisely,

$$\rho_*^{(cl)}(q, p) = \rho_*(H_W(q, p), P_{W1}(q, p), \ldots, P_{WN}(q, p)) \geq 0 \tag{3.3}$$

To simplify the demonstration let us only consider the constant $H_W(q, p)$ and only the continuous spectrum. From Eq. (2.13) we have

$$\rho_* = \int \rho_*(\omega)(\omega|\, d\omega \tag{3.4}$$

So we must compute

$$\rho_\omega^{(cl)}(q, p) = \rho_{W,\omega}(q, p) = \pi^{-1} \int (\omega\|q + \lambda\rangle\langle q - \lambda|)\, e^{2ip\lambda}\, d\lambda \tag{3.5}$$

We know from ref. 1, Section 2.3, that the characteristic property of $(\omega|$ is[11]

$$(\omega|H^n) = \omega^n \tag{3.6}$$

for $n = 0, 1, 2, \ldots$. Using well-known relations between quantum and classical inner products of operators [17], in the limit $\hbar \to 0$ (we will consider that we always take this limit when we refer to classical equations below) we can deduce that the characteristic property of $\rho_\omega^{(cl)}(q, p)$ is

$$\int \rho_\omega^{(cl)}(q, p)[H_W(q, p)]^n\, dq\, dp = \omega^n \tag{3.7}$$

for any natural number n. Thus $\rho_\omega^{(cl)}(q, p)$ must be

[11] This characteristic property of the functional $(\omega]$ is also a property of any usual bra, like the eigenvectors related to discrete eigenvalues.

$$\rho_{\omega}^{(cl)}(q, p) = \delta(H_W(q, p) - \omega) \tag{3.8}$$

in the sense of functional rigorously defined at the beginning of this section. Therefore, going back to Eq. (3.4) and since the Wigner relation is linear, we have

$$\rho_{*}^{(cl)}(q, p) = \int \rho_{*}(\omega)\rho_{\omega}^{(cl)}(q, p)\, d\omega = \int \rho_{*}(\omega)\delta(H_W(q, p) - \omega)\, d\omega$$
$$= \rho_{*}(H_W(q, p)) \geq 0 \tag{3.9}$$

QED

Generalizing this reasoning, we can prove Eq. (3.3). Moreover, the generalized equation (3.9) reads

$$\rho_{*}^{(cl)}(q, p) = \sum_{n, l_1, \ldots, l_N} \rho_{*}(n, l_1, \ldots, l_N)\, \rho_{n, l_1, \ldots, l_N}^{(cl)}(q, p) \geq 0 \tag{3.10}$$

where we give the collective name n to both ω_0 and ω, so the symbol Σ is the usual combination of an integral in ω and an infinite sum of Eq. (2.21); $\rho_{*}(n, l_1, \ldots, l_N) = \rho_{l_1, \ldots, l_N}^{(n)} \geq 0$ and $\rho_{n, l_1, \ldots, l_N}^{(cl)}(q, p)$ reads

$$\rho_{n, l_1, \ldots, l_N}^{(cl)}(q, p) = \delta(H_W(q, p) - n)\, \delta(P_{W1}(q, p) - l_1)$$
$$\ldots \delta(P_{WN}(q, p) - l_N) \tag{3.11}$$

and it can be interpreted as the state that has $n, l_1, \ldots, l_N$ well defined and the corresponding classical canonically conjugated variables completely undefined since $\rho_{n, l_1, \ldots, l_N}^{(cl)}(q, p)$ is not a function of these variables. *So we reach, in the classical case, the same conclusion as in the quantum case (see end of Section 2.2).* But now the all classical canonically conjugate variables $-a_0, a_1, \ldots, a_N$ do exist since they can be found by solving the corresponding Poisson bracket differential equations.

As the momenta $H_W, P_{W1}, \ldots, P_{WN}$, or any function of these momenta, which we will generically call P, is also constant of the motion, then we have $P^{\cdot} = -\partial H/\partial a = 0$, where a is the classically conjugate variable to P. So H is just a function of the P and

$$a^{\cdot} = \frac{\partial H(P)}{\partial P} = \varpi(P) = \text{const} \tag{3.12}$$

so

$$a = \varpi(P)t + a^{(0)} \tag{3.13}$$

Then if we call $-a_0, a_1, \ldots, a_N$ the classical canonically dynamical variables of the momenta $H_W, P_{W1}, \ldots, P_{WN}$, the last equation reads

$$a_0 = t, \qquad a_1 = \varpi_1 t + a_1^{(0)} = \text{const}, \ldots, \qquad a_N = \varpi_N t + a_N^{(0)} = \text{const} \tag{3.14}$$

Thus in the set of classical motions contained in the density (3.11) the momenta $H, P_1, \ldots, P_N$ are completely defined and the origin of the corresponding motions that we have respectively called $a_1^{(0)}, \ldots, a_N^{(0)}$ are completely undefined, in such a way that the motions represented in the last equation homogeneously fill the surface $H, P_1, \ldots, P_N = \text{const}$, which now turn out to be a usual torus of phase space.[12] This is the interpretation that we give to the density (3.11), which turns out to be just a function of the $H, P_1, \ldots, P_N$, and not of the classical conjugate variables $-t, a_1, \ldots, a_N$.

Then, Eq. (3.10) can be considered as the expansion of $\rho_*^{(\mathrm{cl})}(q, p)$ in the sets of classical motions just described, contained in $\rho_{n,l_1,\ldots,l_N}^{(\mathrm{cl})}(q, p)$, each one with a probability $\rho_*(n, l_1, \ldots, l_N)$.

Summing up:

(i) We have shown that the density matrix $\rho(t)$ evolves to a diagonal density matrix ρ_*.

(ii) This density matrix ρ_* has $\rho_*^{(\mathrm{cl})}(q, p)$ as its corresponding classical density.

(iii) This classical density can be decomposed into sets of classical motions where $H, P_1, \ldots, P_N$ remain constant. These motion have origins $a_1^{(0)}, \ldots, a_N^{(0)}$ distributed homogeneously.

(iv) From Eqs. (3.10) and (3.11), Eq. (3.3) is demonstrated and since $\rho(n, l_1, \ldots, l_N) \geq 0$, we also have $\rho_*^{(\mathrm{cl})} \geq 0$.

3.2. Expansion in Single Classical Motions

We can now solve the set of classical motions contained in the density (3.11) in terms of single classical motions. In fact, since

$$\int \prod_{i=1} \delta(a_i - a_i^{(t)}) \prod_{i=1} da_i^{(t)} = 1 \tag{3.15}$$

where

$$a_i^{(t)} = \varpi_i t + a_i^{(0)} \tag{3.16}$$

we can write Eq. (3.10) as

[12] If we consider, for a moment, that $H, P_1, \ldots, P_N$ are isolating constants of the motion.

$$\rho_*^{(\mathrm{cl})}(q,p) = \int \sum_{n,l_1,\ldots,l_N} \rho_*(n,l_1,\ldots,l_N)\,\rho^{(\mathrm{cl})}_{n,l_1,\ldots,l_N}(q,p) \times \prod_{i=1} \delta(a_i - a_i^{(t)}) \prod_{i=1} da_i^{(t)} \tag{3.17}$$

Then if we define

$$\rho^{(\mathrm{cl})}_{n,l_1,\ldots,l_N,a_1^{(0)},\ldots,a_N^{(0)}}(q,p) = \delta(H_W(q,p) - n)\,\delta(P_{W1}(q,p) - l_1) \times \ldots \delta(P_{WN}(q,p) - l_N)\,\delta(a_1 - a_1^{(t)}) \ldots \delta(a_N - a_N^{(t)}) \tag{3.18}$$

which corresponds to the classical distribution of a motion with momenta n, $l_1, \ldots, l_N$ and initial conditions $a_1^{(0)}, \ldots, a_N^{(0)}$, and therefore to a single classical motion. So we can write Eq. (3.17) as

$$\rho_*^{(\mathrm{cl})}(q,p) = \int \sum_{n,l_1,\ldots,l_N} \rho_*(n,l_1,\ldots,l_N)\,\rho^{(\mathrm{cl})}_{n,l_1,\ldots,l_N,a_1^{(t)},\ldots,a_N^{(t)}}(q,p) \prod_{i=1} da_i^{(t)} \tag{3.19}$$

namely Eq. (1.9), as we promised in the introduction.

If for completeness we now restore the primitive notation, i.e., ω_0 and the ω instead of n, and take the integral over the ω, the last equation reads

$$\begin{aligned}\rho_*^{(\mathrm{cl})}(q,p) = \sum_{l_1,\ldots,l_n} [& \int \rho_*(\omega_0,l_1,\ldots,l_N)\,\delta(H-\omega_0)] \\ & \times \prod_{i=1}^{N} \delta(P_i - l_i) \prod_{j=1} \delta(a_j - a_j^{(t)}) \prod_{i=1} da_i^{(t)} \\ & + \int \rho_*(\omega,l_1,\ldots,l_N)\,\delta(H-\omega)] \\ & \times \prod_{i=1}^{N} \delta(P_i - l_i) \prod_{j=1} \delta(a_j - a_j^{(t)}) \prod_{i=1} da_i^{(t)}\,d\omega]\end{aligned} \tag{3.20}$$

So if we write as before

$$\begin{aligned}\rho^{(\mathrm{cl})}_{\omega_0,l_1,\ldots,l_N,a_1^{(t)},\ldots,a_N^{(t)}}(q,p) &= \delta(p_0 - \omega_0)] \prod_{i=1}^{N} \delta(p_i - l_i) \prod_{j=1} \delta(a_j - a_j^{(t)}) \rho^{(\mathrm{cl})}_{\omega,l_1,\ldots,l_N,a_1^{(0)},\ldots,a_N^{(0)}}(q,p) \\ &= \delta(p_0 - \omega)] \prod_{i=1}^{N} \delta(p_i - l_i) \prod_{j=1} \delta(a_j - a_j^{(t)})\end{aligned} \tag{3.21}$$

we have

$$\rho_*^{(\mathrm{cl})}(q,p) = \sum_{l_1,\ldots,l_n} \left[\int \rho_*(\omega_0, l_1, \ldots, l_N) \rho^{(\mathrm{cl})}_{\omega_0, l_1, \ldots, l_N, a_1^{(0)}, \ldots, a_N^{(0)}}(q,p) \prod_{i=1} da_i^{(t)} + \int \rho_*(\omega, l_1, \ldots, l_N) \rho^{(\mathrm{cl})}_{\omega, l_1, \ldots, l_N, a_1^{(0)}, \ldots, a_N^{(0)}}(q,p) \prod_{i=1} da_i^{(t)}\, d\omega \right] \quad (3.22)$$

$\rho^{(\mathrm{cl})}_{\omega_0, l_1, \ldots, l_N, a_1^{(t)}, \ldots, a_N^{(t)}}(q,p)$ and $\rho^{(\mathrm{cl})}_{\omega, l_1, \ldots, l_N, a_1^{(t)}, \ldots, a_N^{(t)}}(x,p)$ are densities that vanish outside of an infinitely narrow tube around the classical trajectory (3.14).

We have obtained the final classical limit. When $t \to \infty$ the quantum density ρ becomes a diagonal density matrix ρ_*. The corresponding classical distribution $\rho_*^{(\mathrm{cl})}(x,p)$ can be expanded as a sum of classical trajectory density functions $\rho^{(\mathrm{cl})}_{\omega_0, l_1, \ldots, l_N, a_1^{(t)}, \ldots, a_N^{(t)}}(x,p)$ and $\rho^{(\mathrm{cl})}_{\omega, l_1, \ldots, l_N, a_1^{(t)}, \ldots, a_N^{(t)}}(x,p)$, each one weighted by its corresponding probability $\rho_*(\omega_0, l_1, \ldots, l_N)$ and $\rho_*(\omega, l_1, \ldots, l_N)$. As the limit of our quantum model we have obtained a statistical classical mechanical model [2] and the classical realm has emerged.

4. CONCLUSION

Using the interplay of observables and states considered as functionals over the space of observables, we have found an *exact final* pointer basis and an *intrinsically* consistent set of final histories. So, given a Hamiltonian H and a state ρ we have found the exact final pointer basis $\{|n, l_1, \ldots, l_N\rangle\}$ and we have shown that $\rho_*^{(\mathrm{cl})}$, the Wigner function of ρ_*, can be expanded in Wigner functions corresponding to the cobasis $(n, l_1, \ldots, l_N|$. To obtain this or similar results almost all authors use coarse-graining methods based on projectors [like the one in Eq. (2.15)] and try to obtain a limit [like the one of Eq. (2.16)]. So they essentially use the weak limit of Eq. (2.12), namely

$$\lim_{t\to\infty} (\rho(t)|O) = (\rho_*|O), \qquad \forall O \in \mathbb{C}$$

But, at least in the classical case, we know that this weak limit exists iff the system is mixing. And the system is mixing iff it has a continuous spectrum [1, 4, 18] and the present paper can be considered as an extension of the theorem that says that the mixing evolutions have a weak limit toward equilibrium [13] in the quantum case. Thus the only way to deal with the problem (at least in the limit $t \to \infty$) in an exact way is to use a method that, like ours, is specially adapted to deal with the singularities inherent in that continuous spectrum. If not, one is condemned to approximate calculations.

Nevertheless, approximate methods are important and, in some cases, unavoidable in order to obtain the non-final pointer basis, but they can be better understood if they are compared with exact methods. We are continuing our research along this line.

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